CHARACTERIZING CONTINUITY IN TOPOLOGICAL SPACES *

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Abstract

Velleman proved that a mapping of $\mathbb{R}$ to $\mathbb{R}$ is continuous iff the images of compact sets are compact and ones of connected sets are connected. Arenas and Puertas generalized this characterization of the continuity to mappings of locally connected first countable spaces to regular normal spaces. We generalize Arenas and Puertas' result.

Introduction.

In [AP] the following definition is given:

Given two topological spaces X and Y, we say that a family $\mathbf{F} \subset Y^X$ can be *characterized by images of sets* if $\mathbf{F} = \mathbf{C}_{\mathbf{A},\mathbf{B}}$ for some families $\mathbf{A} \subset \mathbb{P}(X)$ (note that $\mathbb{P}(X)$ denotes the power set of X) and $\mathbf{B} \subset \mathbb{P}(Y)$, where $\mathbf{C}_{\mathbf{A},\mathbf{B}} = \{f \in Y^X : fA \in \mathbf{B}$ for every $A \in \mathbf{A}\}$.

The problem to characterize the continuity by images of sets was studied first in [V] by Velleman that have shown that the set $C(\mathbb{R}, \mathbb{R})$ of all real continuous functions on $\mathbb{R}$ can not be characterized by images of sets.

In [AP] Arenas and Puertas have demonstrated (generalizing another Velleman's rezult in [V] concerning $C(\mathbb{R}, \mathbb{R})$)(see Abstract) that, under some additional hypothesis, two classes of sets are sufficient to characterize continuity between two topological spaces. In fact, they have proved the following:

THEOREM AP. *Let X be a locally connected first countable space and let Y be a regular normal space. Then*

$$C(X,Y) = \mathbf{C}_{\mathbf{A},\mathbf{A}} \cap \mathbf{C}_{\mathbf{B},\mathbf{B}}$$

where $\mathbf{A}$ is the family of all connected sets and $\mathbf{B}$ is the family of compact sets.

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In the present paper we generalize Theorem AP to some different classes of spaces, by generalizing the definition of set of mappings characterized by images of sets.

Definition 1. Given two topological spaces X and Y and $\mathbf{F} \subset \mathbf{G} \subset Y^X$. We say that $\mathbf{F}$ can be *characterized by images of sets with respect to* $\mathbf{G}$ if $\mathbf{F} = \mathbf{C}^{\mathbf{G}}_{\mathbf{A},\mathbf{B}}$ for some families $\mathbf{A} \subset \mathbb{P}(X)$ and $\mathbf{B} \subset \mathbb{P}(Y)$, where $\mathbf{C}^{\mathbf{G}}_{\mathbf{A},\mathbf{B}} = \{f \in \mathbf{G} : fA \in \mathbf{B}$ for every $A \in \mathbf{A}\}$.

1. Continuity on q-spaces.

Throughout the paper, "space" means topological T_1-space. For terms and undefined concepts we refer to [E].

Definition 2. A sequence $\{O_n\}_{n\in\mathbb{N}}$ of subsets of a space X is called a *q-sequence* in X if for any sequence of points $x_n \in O_n$ (for $n \in \mathbb{N}$) there exists a cluster point (i.e., a point x_0 such that every its neighborhood meets infinitely many x_n's).

Maybe the following lemma is known and its proof is given for reader's convenience.

LEMMA 1. *Let $\{O_n\}_{n\in\mathbb{N}}$ be a q-sequence in a space X. Then:*

(1) if $U_n \subset O_n$ for every $n \in \mathbb{N}$, then the $\{U_n\}_{n\in\mathbb{N}}$ is a q-sequence too;

(2) if $G = \bigcap_{n\in\mathbb{N}} O_n$ is closed, then it is countably compact;

(3) if $G = \bigcap_{n\in\mathbb{N}} O_n$, all O_n are open in X and $clO_{n+1} \subset O_n$ for every $n \in \mathbb{N}$, then G is closed and for any neighborhood O of G there exists n such that $O_n \subset O$;

(4) if $G = \bigcap_{n\in\mathbb{N}} O_n$, all O_n are open in X, $clO_{n+1} \subset O_n$ and the points $x_n \in O_n$, $n \in \mathbb{N}$, are distinct then $H = G \cup \{x_n : n \in \mathbb{N}\}$ is a countably compact set.

Proof. (1) is evident.

(2) Let A be a countably infinite subset of G. If A has no accumulation point in G then every point $x \in G$ has a neighborhood O_x such that $|O_x \cap G| \le 1$. Let f be a 1-1 mapping from $\mathbb{N}$ onto A, then $fn \in O_n$ for every $n \in \mathbb{N}$ but the sequence $\{fn\}_{n\in\mathbb{N}}$ has no cluster point.

(3) Let O be a neighborhood of G. If there exists $x_n \in O_n \setminus O$ for every $n \in \mathbb{N}$ then the sequence $\{x_n\}_{n\in\mathbb{N}}$ has no cluster point (because every O_n contains all but finite elements of the sequence $\{x_n\}_{n\in\mathbb{N}}$ and $X = O \cup \bigcup\{X \setminus clO_n;\ n \in \mathbb{N}\}$).

(4) Let A be a countably infinite subset of H. If A has no accumulation point in G, then every point $x \in G$ has a neighborhood O_x such that $|O_x \cap A| \le 1$ and this intersection is one-point if and only if $x \in A$. Hence, the union $O = \bigcup_{x\in G} O_x$

is a neighborhood of G and, by (2), $O \cap A$ is finite. By (3), there exists n such that $O_n \subset O$. Then $O_n \cap A$ and $A \setminus O_n = A \cap (\{x_m : m \in \mathbb{N}\} \setminus O_n)$ are finite. This is a contradiction. □

Let us recall (see [M]) that a space X is said to be a *q-space* if for every $x \in X$ there exists a q-sequence of neighborhoods of x.

Definition 3. A mapping $f : X \to Y$ between spaces X and Y is said to be *G_δ-continuous* at a point $x \in X$ if, for every neighborhood V of fx, there exists a G_δ-set G in X such that $x \in G$ and $fG \subset V$.
A mapping is called *G_δ-continuous* if it is *G_δ-continuous* at any point of its domain.
The set of all G_δ-continuous mappings from a space X to a space Y will be denoted by $C_\delta(X, Y)$.

Obviously $C(X, Y) \subset C_\delta(X, Y)$ (where $C(X, Y)$ consists of all continuous mappings from X to Y).

Remark. If the space X has countable pseudo-character, any mapping of X is G_δ-continuous (since every singleton of X is a G_δ-set).

THEOREM 2. *Let $f : X \to Y$ be a G_δ-continuous mapping from a locally connected regular q-space X to a normal space Y. If f maps countably compact sets in countably compact sets and connected sets in connected sets, then it is continuous.*

Proof. Suppose, by contradiction, that f is not continuous at some point $x_0 \in X$. Then there exists a neighborhood V of fx_0 such that $fO \setminus V \neq \emptyset$ for every neighborhood O of x_0. By regularity of Y, there exists another neighborhood W of fx_0 such that $clW \subset V$. Furthermore, by using the normality of Y we can obtain a sequence $\{V_n\}_{n \in \mathbb{N}}$ of open sets of Y such that, for every $n \in \mathbb{N}$:

i) $clW \subset V_n \subset clV_n \subset V$ and

ii) $clV_{n+1} \subset V_n$.

Since f is G_δ-continuous, there exists a G_δ-set G of X such that $x_0 \in G$ and $fG \subset W$. Thus, there exists a sequence $\{G_n\}_{n \in \mathbb{N}}$ of open sets of X such that $G = \bigcap_{n \in \mathbb{N}} G_n$. Since X is a q-space, there exists a q-sequence $\{O_n\}_{n \in \mathbb{N}}$ of neighborhoods of x_0.

Furthermore, by using the hypothesis that X is locally connected, it is not restrictive to chose a sequence $\{U_n\}_{n \in \mathbb{N}}$ of open connected sets such that

$$x_0 \in U_1 \subset clU_1 \subset G_1 \cap O_1$$

and that, for every $n > 1$,

$$x_0 \in U_n \subset clU_n \subset G_n \cap \bigcap_{i=1}^{n} O_i \cap \bigcap_{i=1}^{n-1} U_i$$

Hence, $H = \bigcap_{n\in\mathbb{N}} U_n = \bigcap_{n\in\mathbb{N}} clU_n$ and so, H is a closed G_δ-set of X.

Obviously $x_0 \in H \subset G$ and so $fH \subset W$ and, by hypothesis, every image fU_n is a connected set.

By the choose of V, we can find $y_n \in fU_n \setminus V$ (with $n \in \mathbb{N}$). So, for every $n \in \mathbb{N}$, there is some $x_n \in U_n$ such that $fx_n = y_n$.

Since $y_n \in fU_n \setminus V \subset fU_n \setminus clV_n = fU_n \cap (Y \setminus clV_n)$ and $fx_0 \in W \subset clW \subset V_n$, it follows that the connected set fU_n has non empty intersection both with the open set V_n and $Y \setminus V_n$ and hence that $fU_n \cap bdV_n \neq \emptyset$.

Let $z_n \in U_n$ be such that $fz_n \in bdV_n$.

Since, by construction, the family $\{bdV_n\}_{n\in\mathbb{N}}$ is disjoint, the points fz_n and so the points z_n (for $n \in \mathbb{N}$) are distinct. Since $z_n \in U_n \subset O_n$, there exists a point $p \in X$ such that every neighborhood of p contains infinitely many points of $\{z_n\}_{n\in\mathbb{N}}$. Evidently, $p \in \bigcap\{V_n : n \in \mathbb{N}\} = H$.

By Lemma 1, the set $C = H \cup \{z_n\}_{n\in\mathbb{N}}$ is countably compact and, by the hypothesis, the set fC is countably compact too.

But, for every $n \in \mathbb{N}$, we have $fz_n \in bdV_n = clV_n \setminus V_n \subset clV_n \setminus W$ and so no point of fH does not belong to the closure of the set $S = \{fz_n : n \in \mathbb{N}\}$.

Thus S is a closed subset of fC and so S is countably compact.

But, for every $n \in \mathbb{N}$, the point fz_n is isolated in S. In fact, $fz_n \in bdV_n = clV_n \setminus V_n \subset V_{n-1} \setminus clV_{n+1}$. Hence the open set $T = V_{n-1} \setminus clV_{n+1}$ doesn't meet both fz_i with $i = 0, 1 \ldots n-1$ (because they do not belong to V_{n-1}) and fz_i with $i \geq n+1$ because they belong to clV_{n+1}, i.e. $T \cap S = \{fz_n\}$.

Thus S is an infinite discrete space and so it can not be countably compact. □

COROLLARY 3. *A G_δ-continuous mapping from a locally connected regular q-space to a normal space is continuous if and only if it maps countably compact sets in countably compact sets and connected sets in connected sets.*

The previous corollary can be reformulated in the following way:

If X is a locally connected regular q-space and Y is a normal space, then

$$C(X,Y) = \mathbf{C}^{\mathbf{G}}_{\mathbf{A},\mathbf{A}} \cap \mathbf{C}^{\mathbf{G}}_{\mathbf{B},\mathbf{B}} = \mathbf{C}_{\mathbf{A},\mathbf{A}} \cap \mathbf{C}_{\mathbf{B},\mathbf{B}} \cap C_\delta(X,Y)$$

where $\mathbf{G} = C_\delta(X,Y)$, $\mathbf{A}$ is the class of countably compact sets, $\mathbf{B}$ is the class of connected sets.

2. Continuity on sequential spaces.

LEMMA 4. *Every mapping $f : \omega_0 + 1 \to Y$ from the simplest infinite compact space $\omega_0 + 1$ to a space Y that maps infinite compact sets in infinite Hausdorff compact sets is continuous.*

Proof. Suppose, by contradiction, that $f : \omega_0 + 1 \to Y$ is not continuous. Since every point $n \in \omega_0$ is isolated, it necessarily follows that f is not continuous at the point ω_0. So, there exist a neighborhood O of $f\omega_0$ and an infinite set $K \subset \omega_0$ such that $fK \cap O = \emptyset$. The set $K \cup \{\omega_0\}$ is infinite compact and, by hypothesis, its image $f(K \cup \{\omega_0\})$ is Hausdorff infinite compact too. Hence, the set $fK = f(K \cup \{\omega_0\}) \setminus O$ is also Hausdorff compact and infinite. Let y_0 be a cluster point of fK. Then $fK \setminus \{y_0\}$ is infinite and non-closed subset of fK and $K' = K \setminus f^{-1}y_0$ is infinite and so $K' \cup \{\omega_0\}$ is infinite compact. So, by hypothesis, $C = fK' \cup \{f\omega_0\}$ is infinite compact too. Since $fK' \cap O = \emptyset$, the space fK' is closed in C and so is compact. It follows from this that fK' must be closed in the Hausdorff space fK. A contradiction which proves our lemma. □

As consequence of the previous Lemma, we have the following:

THEOREM 5. *A mapping from a sequential space to a Hausdorff space (even to a space in which all countable compact subspaces are Hausdorff) is continuous if it maps infinite countable and compact sets in infinite compact sets.*

Proof. By Lemma 4, restrictions of f to subspaces of X homeomorphic to $\omega_0 + 1$ are continuous. Since X is sequential, f is continuous. □

COROLLARY 6. *A mapping from a sequential space to a Hausdorff space (even to a space in which all countable compact subspaces are Hausdorff) which fibres are discrete is continuous if and only if it maps infinite countable and compact sets in infinite countable and compact sets.*

Proof. If f is continuous then images of all countable infinite compact sets are compact and they can not be finite because no fibre of f does not contain $\omega_0 + 1$. The second part of the corollary follows from Theorem 5. □

COROLLARY 7. *Let X be a sequential space, Y a Hausdorff space and $D(X, Y)$ consist of all mappings of X to Y which fibres are discrete. Then*

$$C(X, Y) \cap D(X, Y) = \mathbf{C}_{\mathbf{A},\mathbf{A}} \cap D(X, Y),$$

where $\mathbf{A}$ is the class of infinite countable compact sets.

COROLLARY 8. *A one-to-one (even injective) mapping from a sequential space to a Hausdorff space (even to a space in which all countable compact subspaces are Hausdorff) is continuous if and only if it maps countable and compact sets in compact sets.*

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